

BOUNDS FOR THE HILBERT TRANSFORM WITH MATRIX A_2 WEIGHTS

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ABSTRACT. Let W denote a matrix A_2 weight. In this paper, we implement a scalar argument using the square function to deduce related bounds for vector-valued functions in $L^2(W)$. These results are then used to study the boundedness of the Hilbert transform and Haar multipliers on $L^2(W)$. Our proof shortens the original argument by Treil and Volberg and improves the dependence on the A_2 characteristic. In particular, we prove that:

$$\|T\|_{L^2(W) \rightarrow L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2},$$

where T is either the Hilbert transform or a Haar multiplier.

1. INTRODUCTION

1.1. Scalar Setting. In this paper, we study the behavior of the Hilbert transform

$$Hf(x) \equiv p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

on matrix-weighted L^2 spaces. To set the scene, recall that in the scalar setting, the Hunt-Muckenhoupt-Wheeden theorem says that for $1 < p < \infty$, the Hilbert transform H is bounded on the weighted space $L^p(w)$ if and only if w is in the A_p Muckenhoupt class, namely, iff

$$(1) \quad [w]_{A_p} \equiv \sup_I \langle w \rangle_I \left\langle w^{-\frac{p'}{p}} \right\rangle_I^{\frac{p}{p'}} < \infty,$$

where the supremum is taken over all intervals I , $\langle w \rangle_I$ denotes the average $\frac{1}{|I|} \int w(x) dx$, and $\frac{1}{p} + \frac{1}{p'} = 1$. More generally, a Calderón-Zygmund operator T is bounded on $L^p(w)$ as long as $w \in A_p$, for $1 < p < \infty$. A subtle, related question that became important to the harmonic analysis community over the past decade is:

What is the dependence of $\|T\|_{L^p(w) \rightarrow L^p(w)}$ on $[w]_{A_p}$?

For $p = 2$, the conjectured dependence was linear, and the problem was termed the A_2 Conjecture. This was resolved by Wittwer for the Haar multipliers [43], by Petermichl-Volberg for the Beurling transform [32], and by Petermichl for the Hilbert transform [28]. The case of general dyadic shifts was handled first by Petermichl implicitly in [29] and then by Lacey-Petermichl-Reguera [21], using different arguments. Finally, in [12], Hytönen

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proved the conjecture for general Calderón-Zygmund operators. Using the sharp form of Rubio de Francia's extrapolation theorem due to Dragičević-Grafakos-Pereyra-Petermichl [7], one can use the linear $L^2(w)$ bound to immediately obtain the sharp result

$$\|T\|_{L^p(w) \rightarrow L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \quad 1 < p < \infty,$$

where the implied constant depends only on T , not the weight w .

1.2. Vector Setting. We are interested in generalizations of this theory to vector-valued functions. Write $L^2 \equiv L^2(\mathbb{R}, \mathbb{C}^d)$, namely those vector-valued functions satisfying

$$\|f\|_{L^2}^2 \equiv \int_{\mathbb{R}} \|f(x)\|_{\mathbb{C}^d}^2 dx < \infty.$$

We say a $d \times d$ matrix-valued function W is a *matrix weight* if W has locally integrable entries and $W(x)$ is positive semidefinite for a.e. x . Then one can define $L^2(W) \equiv L^2(\mathbb{R}, W, \mathbb{C}^d)$ to be the set of vector-valued functions satisfying

$$(2) \quad \|f\|_{L^2(W)}^2 \equiv \int_{\mathbb{R}} \|W^{\frac{1}{2}}(x)f(x)\|_{\mathbb{C}^d}^2 dx = \int_{\mathbb{R}} \langle W(x)f(x), f(x) \rangle_{\mathbb{C}^d} dx < \infty.$$

As defined by Treil-Volberg [41], we say a weight W satisfies the matrix A_2 Muckenhoupt condition if

$$(3) \quad [W]_{A_2} \equiv \sup_I \left\| \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 < \infty,$$

where $\|\cdot\|$ denotes the norm of the matrix acting on \mathbb{C}^d . One can also define $L^p(W)$ and the A_p Muckenhoupt weights. However, for $p \neq 2$, the A_p classes do not have as simple a definition as in (1). Arguably the simplest characterization appears in [37], where Roudenko showed that $W \in A_p$ if and only if

$$\sup_I \frac{1}{|I|} \int_I \left(\frac{1}{|I|} \int_I \left\| W(x)^{\frac{1}{p}} W(t)^{-\frac{1}{p}} \right\|^{p'} dt \right)^{\frac{p}{p'}} dx < \infty.$$

For additional details and characterizations of A_p weights, we refer the readers to [10, 22, 26, 37, 42].

Treil-Volberg chose to characterize A_2 weights as ones satisfying (3) because in [41], they proved: the Hilbert transform is bounded on $L^2(W)$ if and only if W satisfies (3). They gave an alternate proof in [40]. In [24, 42], Nazarov-Treil and Volberg separately generalized this result to A_p weights. They both also showed that a classical Calderón-Zygmund operator is bounded on $L^p(W)$ if W is in A_p . Here, “classical” means that the operator is defined by applying a scalar Calderón-Zygmund operator T to each component of a vector-valued function and further, T satisfies $T1 = T^*1 = 0$. In [5, 10], Christ-Goldberg and Goldberg studied a class of weighted, vector analogues of the Hardy-Littlewood maximal function and used them to establish the boundedness of a class of singular integral operators on $L^p(W)$, for $W \in A_p$.

A host of related interesting problems have also been examined in the matrix setting. For instance, in [1, 16, 17], Bickel-Wick and Kerr established $T(1)$ theorems characterizing the boundedness of operators, including the Hilbert transform, between matrix weighted spaces. Meanwhile, in [18], Isralowitz-Kwon-Pott studied the boundedness of commutators of the form $[T, B]$ on $L^p(W)$, where T is a Riesz transform and B a locally integrable

matrix function. In [11, 14, 23, 25], Nazarov-Treil-Volberg, Katz, Goldberg, and Nazarov-Pisier-Treil-Volberg studied the dependence of the unweighted Carleson Embedding Theorem in the matrix setting on dimension d and concluded that its sharp dependence is $\log d$. Similarly, in [25, 27] Nazarov-Pisier-Treil-Volberg and Petermichl studied vector-valued Hankel operators, again concluding that the operator norm's sharp dependence on dimension is $\log d$.

In the operator weighted setting, less is known. Gillespie-Pott-Treil-Volberg studied the Haar multipliers and Hilbert transform on weighted spaces in [8, 9]. They showed W satisfying (3) no longer implies that the Hilbert transform or Haar multipliers are bounded on $L^2(W)$. In [31], Petermichl-Pott proved a form of Burkholder's Theorem, connecting the boundedness of the Haar multipliers with that of the Hilbert transform on operator weighted L^2 spaces. In [15, 33], Pott and Katz-Pereyra both provided interesting sufficient conditions for the Hilbert transform to be bounded on operator weighted L^2 , but to the best of the authors' knowledge, necessary and sufficient conditions have proved elusive.

1.3. Matrix A_2 Conjecture. In this paper, we are interested in the natural sharpness question. For W an A_2 matrix weight and T a Calderón-Zygmund operator,

How does $\|T\|_{L^2(W) \rightarrow L^2(W)}$ depend on $[W]_{A_2}$?

In analogy with the scalar setting, we conjecture that the dependence is linear. However, very little is actually known about the answer, and the sharp bounds for even "simple" operators like the Hilbert transform and Haar multipliers are unknown. Currently, the best known results concern sparse operators. In [2, 18], Bickel-Wick and Isralowitz-Kwon-Pott separately established that if \mathcal{S} is a sparse operator, then

$$\|\mathcal{S}\|_{L^2(W) \rightarrow L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}}.$$

Similarly, in [18], Isralowitz-Kwon-Pott studied Christ-Goldberg's maximal function and showed that on $L^2(W)$, its norm depends linearly on $[W]_{A_2}$.

In this vector setting, progress is slow because many tools in the scalar case do not exist or generalize poorly to the matrix setting. For example, Petermichl used both a bilinear embedding theorem and Bellman function testing conditions to show that the scalar Hilbert transform's norm depends linearly on $[w]_{A_2}$ [28]. In the matrix weighted setting, there is no known sharp bilinear embedding theorem and Bellman function arguments, while possible, are much more difficult to execute. Indeed, many arguments fail because simple scalar facts like $0 < w < v$ implies $w^2 < v^2$ do not hold for matrices.

In this paper, we show that with care, some important scalar arguments can be modified to work in the matrix setting. Specifically, we consider an elegant proof of Petermichl-Pott from [30], establishing the boundedness of the Hilbert transform on $L^2(w)$ with dependence $[w]_{A_2}^{\frac{3}{2}}$. By modifying this argument appropriately, we prove that

$$(4) \quad \|H\|_{L^2(W) \rightarrow L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log[W]_{A_2}$$

and obtain similar results for Haar multipliers. Although these constants do not appear to be sharp, they are better than anything that has previously appeared in the literature. While it seems unlikely that the additional $\log[W]_{A_2}$ is required, removing it will certainly require nontrivial new ideas. We also mention that our results do not extend immediately to $L^p(W)$, as there are complications related to extrapolation in the matrix setting.

1.4. Outline of Paper. In Section 2, we introduce the basic notation and tools used in the proofs of the main results. These tools include sets of disbalanced Haar functions adapted to matrix weights and a weighted matrix embedding theorem. The remainder of the paper discusses the generalization of Petermichl-Pott's result to the matrix setting as well as current conjectures and potential modifications.

In Section 3, we prove preliminary bounds involving a generalized square function, which are interesting in their own right. The main results, given in Theorems 3.1 and 3.2, are the following upper and lower estimates:

$$\sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \lesssim [W]_{A_2}^2 \log [W]_{A_2} \|f\|_{L^2(W)}^2;$$

$$\|f\|_{L^2(W)}^2 \lesssim [W]_{A_2} \log [W]_{A_2} \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d},$$

which hold for all $f \in L^2(W)$. Other proofs of the upper square function bound appear in [19, 24, 42]. Our paper improves the stated dependence of $[W]_{A_2}^4$ and $[W]_{A_2}^3$ in [19]. Similarly, although the [24, 42] proofs will give constants depending on $[W]_{A_2}$, we did not track this dependence explicitly, and it seems unlikely that any resulting constants would be near optimal. It is also worth pointing out that the analogous estimates appearing in [30] for the scalar case do not include a $\log[w]_{A_2}$ term. Rather, this comes from the matrix embedding theorem. Another technicality is that to use the scalar arguments, we needed to reduce to the situation of bounded matrix weights; this reduction is handled in Remark 3.3 and involves truncations at the level of eigenvalues.

In Section 4, we prove the previously-discussed bound for the Hilbert transform (4), which appears in Theorem 4.1. This follows from an estimate on first order Haar shifts, since the Hilbert transform can be represented as an average of such Haar shifts. The main argument uses our previous square function bounds and the fact that the square function norm is unaffected by these simple Haar shifts.

In Section 5, we apply similar arguments to Haar multipliers. Namely, let $\sigma = \{\sigma_I\}_{I \in \mathcal{D}}$ be a sequence of matrices indexed by the dyadic intervals and define the Haar multiplier T_σ by

$$T_\sigma f \equiv \sum_{I \in \mathcal{D}} \sigma_I \widehat{f}(I) h_I,$$

where the Haar coefficients $\widehat{f}(I)$ and Haar functions $\{h_I\}$ are defined precisely in Section 2. Then for all $f \in L^2(W)$, we show in Theorem 5.2 that

$$\|T_\sigma\|_{L^2(W) \rightarrow L^2(W)} \lesssim \|\sigma\|_\infty [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2},$$

where $\|\sigma\|_\infty = \sup_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{\frac{1}{2}} \sigma_I \langle W \rangle_I^{-\frac{1}{2}} \right\|$. In [18], Isralowitz-Kwon-Pott established this boundedness result for $p = 2$ and a similar one for all $1 < p < \infty$. In their paper's most recent version, they also mention that, using our square function bounds in Theorems 3.1 and 3.2, their proof will give the same dependence on $[W]_{A_2}$ in the $p = 2$ case. A recent result by Pott-Stoica [35], which uses methods from [39], reduces the study of Calderón-Zygmund operators to the study of Haar multipliers. Pairing this with our estimate gives

$$\|T\|_{L^2(W) \rightarrow L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2},$$

for all Calderón-Zygmund operators T .

In Section 6, we discuss related open questions and conjectures. Specifically, one would hope to remove the $\log[W]_{A_2}$ from our norm bounds. One original barrier was the lack of a sharp weighted matrix Carleson Embedding Theorem, like the one used in [30]. Very recently, this result was proved by Culiuc-Treil in [6]. In Section 6, we show how to modify our earlier arguments to potentially use this new Carleson Embedding Theorem to improve our bounds on the square function, Hilbert transform, and Haar multipliers. However, finishing the proof requires a testing condition, which so far has remained elusive. Finally, one can ask if there are similar bounds for operators on $L^p(W)$, with the weight W in A_p . This is an interesting but hard problem related to sharp extrapolation. We end our paper with a discussion of the potential complications arising in the matrix setting.

2. BASIC FACTS AND NOTATION

Throughout this paper, $A \lesssim B$ indicates that $A \leq C(d)B$, for some constant $C(d)$ that may depend on the dimension d .

2.1. Dyadic Grids. Let \mathcal{D} denote the standard dyadic grid. For $\alpha \in \mathbb{R}$ and $r > 0$, let $\mathcal{D}^{\alpha,r}$ denote the dyadic grid $\{\alpha + rI : I \in \mathcal{D}\}$ and let $\{h_I\}_{I \in \mathcal{D}^{\alpha,r}}$ denote the Haar functions adapted to $\mathcal{D}^{\alpha,r}$ and normalized in L^2 . We will use these shifted dyadic grids in Section 4, when studying the Hilbert transform. However, in much of what follows, we omit the superscripts α, r because the arguments hold for all such dyadic grids.

To be precise, for $I \in \mathcal{D}$, let I_+ denote its right half and I_- its left half. Then h_I is defined by

$$h_I \equiv |I|^{-\frac{1}{2}} (\mathbf{1}_{I_+} - \mathbf{1}_{I_-}) \quad \forall I \in \mathcal{D},$$

where $\mathbf{1}_E$ is the characteristic function of the set E . Similarly, define $h_I^1 \equiv \mathbf{1}_{I \setminus |I|}$ for any $I \in \mathcal{D}$. One should notice that the non-cancellative Haar functions have a different normalization. Now, let $f \in L^2$. To define $\widehat{f}(I)$, let ν_1, \dots, ν_d be an orthonormal basis in \mathbb{C}^d . Then,

$$\widehat{f}(I) \equiv \int_I f(x) h_I(x) dx = \sum_{j=1}^d \langle f, h_I \nu_j \rangle_{L^2} \nu_j.$$

Note that this decomposition works for *any* orthonormal basis. In the later proofs, we will use an orthonormal basis that depends on I .

2.2. Disbalanced Haar functions. If W is a matrix weight, then its entries are locally-integrable and we can define

$$W(I) \equiv \int_I W(x) dx \text{ and } \langle W \rangle_I \equiv \frac{1}{|I|} \int_I W(x) dx.$$

Similarly, we have:

$$\widehat{W}(I) = \int_{\mathbb{R}} W(x) h_I(x) dx = \frac{1}{2} |I|^{\frac{1}{2}} \left(\langle W \rangle_{I_+} - \langle W \rangle_{I_-} \right).$$

Then $L^2(W)$ is defined by (2) and if $f, g \in L^2(W)$, then

$$\langle f, g \rangle_{L^2(W)} = \int_{\mathbb{R}} \langle W(x) f(x), g(x) \rangle_{\mathbb{C}^d} dx.$$

In the main proof, we will use *disbalanced Haar functions adapted to W* . Treil and Volberg introduce these in the matrix setting in [41]. To define them, fix $I \in \mathcal{D}$ and let e_I^1, \dots, e_I^d be a set of orthonormal eigenvectors of $\langle W \rangle_I$. Define

$$w_I^k \equiv \left\| \langle W \rangle_I^{\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^{-1} = \left\| \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}.$$

Then, the vector-valued functions $\{w_I^k h_I e_I^k\}_{I \in \mathcal{D}, 1 \leq k \leq d}$ are normalized in $L^2(W)$. Define the disbalanced Haar functions

$$g_I^{W,k} \equiv w_I^k h_I e_I^k + h_I^1 \tilde{e}_I^k,$$

where the vector $\tilde{e}_I^k = A(W, I) e_I^k$ and

$$A(W, I) = \frac{1}{2} |I|^{\frac{1}{2}} \langle W \rangle_I^{-1} (\langle W \rangle_{I_-} - \langle W \rangle_{I_+}) \langle W \rangle_I^{-\frac{1}{2}}.$$

Simple calculations, which appear in [41], show that

$$(5) \quad \left\langle g_I^{W,k}, g_J^{W,j} \right\rangle_{L^2(W)} = 0 \quad \forall J \neq I, 1 \leq j, k \leq d,$$

and the functions satisfy $\|g_I^{W,k}\|_{L^2(W)} \leq 5$. It is worth pointing out that for $I = J$, then the inner product

$$\left\langle g_I^{W,k}, g_I^{W,j} \right\rangle_{L^2(W)}$$

need not be zero. Using simple computations, we can write standard Haar functions using these disbalanced ones as follows

$$(6) \quad h_I e_I^k = (w_I^k)^{-1} g_I^{W,k} - (w_I^k)^{-1} A(W, I) h_I^1 e_I^k$$

for all $I \in \mathcal{D}$ and $k = 1, \dots, d$.

2.3. Carleson Embedding Theorem. To prove our main results, we initially proceed as in Petermichl-Pott's proof of the scalar case in [30]. Some arguments generalize easily, but to finish the proof, we require a matrix weighted embedding theorem. Specifically, we use the following result of Treil-Volberg, which appears as Theorem 6.1 in [41]:

Theorem 2.1 (Treil and Volberg, [41]). *Let W be a $d \times d$ matrix weight in A_2 . Then for all $f \in L^2$,*

$$\sum_{I \in \mathcal{D}} |I| \left\| \langle W \rangle_I^{-\frac{1}{2}} (\langle W \rangle_{I_-} - \langle W \rangle_{I_+}) \langle W \rangle_I^{-\frac{1}{2}} \right\|^2 \left\| \langle W \rangle_I^{-\frac{1}{2}} \left\langle W^{\frac{1}{2}} f \right\rangle_I \right\|^2 \lesssim [W]_{A_2} \log [W]_{A_2} \|f\|_{L^2}^2.$$

The constant $[W]_{A_2} \log [W]_{A_2}$ is not specified in Treil-Volberg's statement of the theorem. However, a careful reading of the proofs of their Lemma 3.1, Lemma 3.6, Theorem 4.1, and Theorem 6.1 reveal the above constant.

3. SQUARE FUNCTION ESTIMATE

Recall the dyadic Littlewood-Paley square function, typically defined by

$$(7) \quad S f(x)^2 \equiv \sum_{I \in \mathcal{D}} \left| \widehat{f}(I) \right|^2 h_I^1(x),$$

for f in $L^2(\mathbb{R}, \mathbb{C})$, which coincides with the usual definition summing square norms of martingale differences in the dyadic filtration. Here is an alternate formulation. Let $\{-1, 1\}^{\mathcal{D}}$ denote the set of all sequences indexed by the dyadic intervals whose terms only

take the values ± 1 . Let $\sigma \equiv \{\sigma_I\}_{I \in \mathcal{D}}$ be an element of $\{-1, 1\}^{\mathcal{D}}$ and let T_σ denote the associated Haar multiplier

$$T_\sigma f = \sum_{I \in \mathcal{D}} \sigma_I \widehat{f}(I) h_I.$$

For any fixed x and collection \mathcal{D}_x of dyadic intervals containing x , consider the collection of sequences $\{-1, 1\}^{\mathcal{D}_x}$ interpreted as the probability space of random sequences of independent realisations of a random variable taking values in ± 1 with equal probability. So we associate the natural probability measure that assigns measure 2^{-k} to each cylinder of length k (Bernoulli measure). Then, as mentioned in [30], the square function can be equivalently defined as

$$(8) \quad Sf(x)^2 \equiv \mathbb{E}(|T_\sigma f(x)|^2).$$

This is equivalent to the previous definition (7) because

$$\mathbb{E}(|T_\sigma f(x)|^2) = \sum_{I, J \in \mathcal{D}} \mathbb{E}(\sigma_I \sigma_J) \widehat{f}(I) \overline{\widehat{f}(J)} h_I(x) h_J(x) = \sum_{I \in \mathcal{D}} |\widehat{f}(I)|^2 h_I^1(x).$$

This follows because each σ_I takes values ± 1 with equal probability. Hence, $\mathbb{E}(\sigma_I \sigma_J) = 0$ if $I \neq J$ and $\mathbb{E}(\sigma_I \sigma_J) = 1$ if $I = J$.

3.1. Generalized Square Function. The classical vector analogue of (7) is

$$Sf(x)^2 \equiv \sum_{I \in \mathcal{D}} \left\| \widehat{f}(I) \right\|_{\mathbb{C}^d}^2 h_I^1(x).$$

Here Sf is naturally scalar-valued, as it is equal to the square function summing square norms of martingale differences of vector valued martingales. However, this definition is not useful in the weighted setting because it does not make sense to study S as an operator from $L^2(W)$ to $L^2(W)$. Instead, to incorporate weights, we define a different square function S_W for each weight W . Pulling the weight inside an operator like this is a standard trick when studying boundedness; for instance, instead of studying $T : L^2(w) \rightarrow L^2(w)$, it is often more convenient to study $M_w T : L^2(w) \rightarrow L^2(w^{-1})$, where M_w is multiplication by w . This trick has proven essential in the vector-valued theory. For example, Christ-Goldberg's maximal functions from [5, 10] incorporate the matrix weights into the operator as follows

$$(9) \quad M_W^p f(x) \equiv \sup_{I: x \in I} \frac{1}{|I|} \int_I \left\| W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y) f(y) \right\|_{\mathbb{C}^d} dy, \quad 1 < p < \infty,$$

where the superscript p in M_W^p just indicates the dependence of the operator on p . These maximal operators map into scalar-valued spaces of functions and give important information about A_p weights. For our square function, we do something similar. We still let

$$T_\sigma f = \sum_{I \in \mathcal{D}} \sigma_I \widehat{f}(I) h_I,$$

where σ is an arbitrary sequence in $\{1, -1\}^{\mathcal{D}}$. Then if we define

$$S_W : L^2(\mathbb{R}, \mathbb{C}^d) \rightarrow L^2(\mathbb{R}, \mathbb{R}) \quad \text{by} \quad S_W f(x)^2 \equiv \mathbb{E} \left(\left\| W(x)^{\frac{1}{2}} T_\sigma f(x) \right\|_{\mathbb{C}^d}^2 \right),$$

we have

$$\begin{aligned}
\|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 &= \int_{\mathbb{R}} \mathbb{E} \sum_{I, J \in \mathcal{D}} \sigma_I \sigma_J h_I(x) h_J(x) \left\langle W(x) \widehat{f}(I), \widehat{f}(J) \right\rangle_{\mathbb{C}^d} dx \\
&= \int_{\mathbb{R}} \sum_{I, J \in \mathcal{D}} \mathbb{E}(\sigma_I \sigma_J) h_I(x) h_J(x) \left\langle W(x) \widehat{f}(I), \widehat{f}(J) \right\rangle_{\mathbb{C}^d} dx \\
&= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d}.
\end{aligned}$$

Volberg introduced the study of these sums in [42]. Notice that in the scalar situation, we can similarly define S_w by

$$S_w f(x)^2 \equiv \mathbb{E} \left(|w(x)^{\frac{1}{2}} T_\sigma f(x)|^2 \right).$$

It is immediate that

$$\|S_w f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 = \sum_{I \in \mathcal{D}} \langle w \rangle_I |\widehat{f}(I)|^2 = \|Sf\|_{L^2(w)}^2.$$

So, in the scalar situation, these square functions S_w are bounded from $L^2(w)$ to $L^2(\mathbb{R}, \mathbb{R})$ precisely when the standard square function S is bounded from $L^2(w)$ to $L^2(w)$, and the operators have the same norm. Thus, studying S_W from $L^2(W)$ to $L^2(\mathbb{R}, \mathbb{R})$ gives a natural generalization of the standard square function questions.

3.2. Square Function Bounds. In the scalar setting, the square function S is bounded on $L^2(w)$ if and only if w is an A_2 weight and the dependence on $[w]_{A_2}$ is linear. For matrix A_2 weights and the new square functions S_W , we obtain the following similar bound, which differs from the scalar bound by a logarithm:

Theorem 3.1. *Let W be a $d \times d$ matrix weight in A_2 . Then*

$$\|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \lesssim [W]_{A_2}^2 \log [W]_{A_2} \|f\|_{L^2(W)}^2 \quad \forall f \in L^2(W).$$

To establish Theorem 3.1, we follow the arguments in [30], which first prove a lower bound on the square function. Our matrix analogue is Theorem 3.2 and the proof uses both arguments from [30] and Theorem 2.1. As with Theorem 3.1, this lower bound differs from the scalar bound by a factor of $\log [W]_{A_2}$.

Theorem 3.2. *Let W be a $d \times d$ matrix weight in A_2 . Then*

$$\|f\|_{L^2(W)}^2 \lesssim [W]_{A_2} \log [W]_{A_2} \|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \quad \forall f \in L^2(W).$$

Proof. As in [30], we can assume without loss of generality that W and W^{-1} are bounded. For more details, see Remark 3.3. Then $L^2(W)$ and L^2 are equal as sets. For ease of notation, define the constant

$$C_W \equiv [W]_{A_2} \log [W]_{A_2}.$$

Let e_1, \dots, e_d be the standard orthonormal basis in \mathbb{C}^d . Define the discrete multiplication operator $D_W : L^2 \rightarrow L^2$ by

$$D_W : h_I e_k \mapsto \langle W \rangle_I h_I e_k \quad \forall I \in \mathcal{D}, k = 1, \dots, d.$$

Observe that

$$\langle D_W f, f \rangle_{L^2} = \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d}.$$

Let M_W denote multiplication by W . Then, we can rewrite the desired inequality as

$$(10) \quad \langle M_W f, f \rangle_{L^2} \lesssim C_W \langle D_W f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

As in [30], we convert this to an inverse inequality. Since W and W^{-1} are bounded, D_W and M_W are bounded and invertible with $M_W^{-1} = M_{W^{-1}}$ and D_W^{-1} defined by

$$D_W^{-1} : h_I e_k \mapsto \langle W \rangle_I^{-1} h_I e_k \quad \forall I \in \mathcal{D}, k = 1, \dots, d.$$

M_W and D_W and their inverses have well-defined square roots, with $M_W^{\frac{1}{2}} = M_{W^{\frac{1}{2}}}$ and $D_W^{\frac{1}{2}}$ sending each $h_I e_k$ to $\langle W \rangle_I^{\frac{1}{2}} h_I e_k$. Similarly, the spectral theorem implies that the positive, invertible, self-adjoint operator $D_W^{-\frac{1}{2}} M_W D_W^{-\frac{1}{2}}$ has a positive, invertible square root. Then, (10) is immediately equivalent to

$$\langle D_W^{-\frac{1}{2}} M_W D_W^{-\frac{1}{2}} f, f \rangle_{L^2} \lesssim C_W \langle f, f \rangle_{L^2}, \quad \forall f \in L^2,$$

which one can show is equivalent to

$$(11) \quad \langle D_W^{-1} f, f \rangle_{L^2} \lesssim C_W \langle M_W^{-1} f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

So to prove Theorem 3.2, we need to establish:

$$\sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{-1} \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \lesssim C_W \|f\|_{L^2(W^{-1})}^2 \quad \forall f \in L^2.$$

We will rewrite the sum using Haar functions adapted to W . First, for $I \in \mathcal{D}$, let e_I^1, \dots, e_I^d be a set of orthonormal eigenvectors of $\langle W \rangle_I$. Recall that

$$w_I^k \equiv \left\| \langle W \rangle_I^{\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^{-1} = \left\| \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d},$$

so w_I^k is the reciprocal of the square root of the eigenvalue corresponding to eigenvector e_I^k . Using these definitions, expand the sum as follows:

$$\begin{aligned} \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{-1} \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} &= \sum_{I \in \mathcal{D}} \sum_{j,k=1}^d \left\langle \langle W \rangle_I^{-1} \langle f, h_I e_I^k \rangle_{L^2} e_I^k, \langle f, h_I e_I^j \rangle_{L^2} e_I^j \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \sum_{j,k=1}^d \langle f, h_I e_I^k \rangle_{L^2} \overline{\langle f, h_I e_I^j \rangle_{L^2}} \left\langle \langle W \rangle_I^{-1} e_I^k, e_I^j \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle f, h_I e_I^k \rangle_{L^2} \right|^2 \left\langle \langle W \rangle_I^{-1} e_I^k, e_I^k \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \sum_{k=1}^d (w_I^k)^2 \left| \langle f, h_I e_I^k \rangle_{L^2} \right|^2. \end{aligned}$$

Now, we can expand the $h_I e_I^k$ using the disbalanced Haar functions adapted to W as in (6). This transforms our sum as follows:

$$\begin{aligned}
\sum_{I \in \mathcal{D}} \sum_{k=1}^d (w_I^k)^2 |\langle f, h_I e_I^k \rangle_{L^2}|^2 &= \sum_{I \in \mathcal{D}} \sum_{k=1}^d (w_I^k)^2 \left| \langle f, (w_I^k)^{-1} g_I^{W,k} - (w_I^k)^{-1} A(W, I) h_I^1 e_I^k \rangle_{L^2} \right|^2 \\
&\leq \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle f, g_I^{W,k} \rangle_{L^2} \right|^2 \\
&\quad + 2 \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle f, g_I^{W,k} \rangle_{L^2} \langle f, A(W, I) h_I^1 e_I^k \rangle_{L^2} \right| \\
&\quad + \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle f, A(W, I) h_I^1 e_I^k \rangle_{L^2} \right|^2 \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

It is clear that

$$S_1 = \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle f, g_I^{W,k} \rangle_{L^2} \right|^2 = \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \langle W^{-1} f, g_I^{W,k} \rangle_{L^2(W)} \right|^2 \lesssim \|f\|_{L^2(W^{-1})}^2,$$

since the $\{g_I^{W,k}\}$ satisfy (5) and are uniformly bounded in $L^2(W)$. Since $S_2 \lesssim S_1^{\frac{1}{2}} S_3^{\frac{1}{2}}$, the main term to understand is S_3 . It can be written as

$$(12) \quad S_3 = \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \left\langle f, \frac{1}{2} |I|^{\frac{1}{2}} \langle W \rangle_I^{-1} (\langle W \rangle_{I_-} - \langle W \rangle_{I_+}) \langle W \rangle_I^{-\frac{1}{2}} h_I^1 e_I^k \right\rangle_{L^2} \right|^2$$

$$= \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \left\langle f, \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-\frac{1}{2}} h_I^1 e_I^k \right\rangle_{L^2} \right|^2$$

$$= \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \left\langle \langle f \rangle_I, \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\rangle_{\mathbb{C}^d} \right|^2$$

$$(13) \quad = \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \left\langle \langle W \rangle_I^{-\frac{1}{2}} \langle f \rangle_I, \langle W \rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\rangle_{\mathbb{C}^d} \right|^2.$$

Now, we can bound S_3 as follows:

$$\begin{aligned}
S_3 &\leq \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle f \rangle_I \right\|_{\mathbb{C}^d}^2 \left\| \langle W \rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W \rangle_I^{-\frac{1}{2}} e_I^k \right\|_{\mathbb{C}^d}^2 \\
&\lesssim \sum_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{-\frac{1}{2}} \langle f \rangle_I \right\|_{\mathbb{C}^d}^2 \left\| \langle W \rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W \rangle_I^{-\frac{1}{2}} \right\|^2 \\
&\lesssim [W]_{A_2} \log [W]_{A_2} \|f\|_{L^2(W^{-1})}^2,
\end{aligned}$$

where we used Theorem 2.1 applied to $g = W^{-\frac{1}{2}} f$. This also implies a similar bound for S_2 , and combining our estimates for S_1, S_2, S_3 completes the proof of Theorem 3.2. \square

Using Theorem 3.2, we can easily prove Theorem 3.1:

Proof. Again, assume without loss of generality that W and W^{-1} are bounded and define the constant B_W by

$$B_W = [W]_{A_2}^2 \log [W]_{A_2} = [W]_{A_2} C_W.$$

Using our previous notation, Theorem 3.1 is equivalent to the inequality

$$\langle D_W f, f \rangle_{L^2} \lesssim B_W \langle M_W f, f \rangle_{L^2}, \quad \forall f \in L^2.$$

We require the following operator inequality

$$D_W \leq [W]_{A_2} (D_{W^{-1}})^{-1}.$$

The A_2 condition implies that for every $I \in \mathcal{D}$ and vector $e_I \in \mathbb{C}^d$,

$$\left\langle \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} e_I, \langle W \rangle_I^{\frac{1}{2}} \langle W^{-1} \rangle_I^{\frac{1}{2}} e_I \right\rangle_{\mathbb{C}^d} \leq [W]_{A_2} \|e_I\|_{\mathbb{C}^d}^2.$$

Fixing $g \in L^2$ and setting $e_I = \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I)$, we can conclude

$$\left\langle \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I), \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \leq [W]_{A_2} \left\langle \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I), \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d}.$$

Then

$$\begin{aligned} \langle D_W g, g \rangle_{L^2} &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I), \langle W \rangle_I^{\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \\ &\leq [W]_{A_2} \sum_{I \in \mathcal{D}} \left\langle \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I), \langle W^{-1} \rangle_I^{-\frac{1}{2}} \widehat{g}(I) \right\rangle_{\mathbb{C}^d} \\ &= [W]_{A_2} \langle (D_{W^{-1}})^{-1} g, g \rangle_{L^2}. \end{aligned}$$

Combining that estimate with (11) from Theorem 3.2 applied to W^{-1} gives:

$$\langle D_W g, g \rangle_{L^2} \leq [W]_{A_2} \langle (D_{W^{-1}})^{-1} g, g \rangle_{L^2} \lesssim [W]_{A_2} C_W \langle M_{W^{-1}}^{-1} g, g \rangle_{L^2} = B_W \|g\|_{L^2(W)}$$

for all $g \in L^2$, which completes the proof. \square

Remark 3.3 (Reducing to Bounded Weights). The proofs of Theorems 3.1 and 3.2 only handle weights W with both W and W^{-1} bounded. To reduce to this case, fix $W \in A_2$ and write

$$W(x) = \sum_{j=1}^d \lambda_j(x) P_{E_j(x)} \quad \text{for } x \in \mathbb{R},$$

where the $\lambda_j(x)$ are eigenvalues of $W(x)$, the $E_j(x)$ are the associated orthogonal eigenspaces, and the $P_{E_j(x)}$ are the orthogonal projections onto the $E_j(x)$. Define

$$\begin{aligned} E_1^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \lambda_j(x) \leq \frac{1}{n}; \\ E_2^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \frac{1}{n} < \lambda_j(x) < n; \\ E_3^n(x) &\equiv \text{Eigenspaces of } W(x) \text{ corresponding to eigenvalues } \lambda_j(x) \geq n. \end{aligned}$$

Using these spaces, truncate $W(x)$ as follows:

$$W_n(x) = \frac{1}{n} P_{E_1^n(x)} + P_{E_2^n(x)} W(x) P_{E_2^n(x)} + n P_{E_3^n(x)}.$$

It is immediate that

$$(W_n(x))^{-1} = n P_{E_1^n(x)} + P_{E_2^n(x)} W^{-1}(x) P_{E_2^n(x)} + \frac{1}{n} P_{E_3^n(x)}.$$

It is easy to see that $\frac{1}{n}I_{d \times d} \leq W_n, W_n^{-1} \leq nI_{d \times d}$. Each W_n is also an A_2 weight with

$$(14) \quad [W_n]_{A_2} \equiv \sup_I \left\| \langle W_n \rangle_I^{\frac{1}{2}} \langle W_n^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 \lesssim [W]_{A_2},$$

where the constant depends on the dimension d . For the scalar case, in [36], Reznikov-Vasyunin-Volberg show that the constant in (14) is one. So, it would be interesting to determine the best constant in (14).

Our proof of (14) relies on the following two facts about positive self-adjoint matrices:

Fact 1: If $A_1, A_2 \geq 0$, then $\left\| A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \right\|^2 \approx \text{Tr}(A_1 A_2)$.

Fact 2: If $A_1, A_2, B_1, B_2 \geq 0$ and each $A_j \leq B_j$, then $\text{Tr}(A_1 A_2) \leq \text{Tr}(B_1 B_2)$.

Here, the implied constants again depend on d . Fact 1 allows us to equate $\left\| \langle W_n \rangle_I^{\frac{1}{2}} \langle W_n^{-1} \rangle_I^{\frac{1}{2}} \right\|^2 \approx \text{Tr}(\langle W_n \rangle_I \langle W_n^{-1} \rangle_I)$. Then, using Fact 2 and the matrix inequalities

$$\begin{aligned} \langle W_n \rangle_I &\leq nI_{d \times d}; \\ \langle P_{E_2^n(x)} W(x) P_{E_2^n(x)} + nP_{E_3^n(x)} \rangle_I &\leq \langle W \rangle_I, \end{aligned}$$

for W_n and similar ones for W_n^{-1} , one can deduce that

$$\text{Tr}(\langle W_n \rangle_I \langle W_n^{-1} \rangle_I) \leq 2 \text{Tr}(I_{d \times d}) + \text{Tr}(\langle W \rangle_I \langle W^{-1} \rangle_I).$$

Applying Fact 1 and using $1 \leq [W]_{A_2}$ immediately gives (14). Then, as W_n and W_n^{-1} are bounded, the arguments in the proof of Theorem 3.2 imply that

$$(15) \quad \|f\|_{L^2(W_n)}^2 \lesssim [W_n]_{A_2} \log [W_n]_{A_2} \sum_{I \in \mathcal{D}} \left\langle \langle W_n \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} \quad \forall f \in L^2(W_n).$$

Using basic convergence theorems, we will see that both

$$(16) \quad \lim_{n \rightarrow \infty} \|f\|_{L^2(W_n)}^2 = \|f\|_{L^2(W)}^2$$

and

$$(17) \quad \lim_{n \rightarrow \infty} \sum_{I \in \mathcal{D}} \left\langle \langle W_n \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} = \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d},$$

for $f \in L^2 \cap L^2(W)$. In fact, to obtain the first inequality, observe that

$$\langle W_n(x)f(x), f(x) \rangle_{\mathbb{C}^d} \leq \langle f(x) + W(x)f(x), f(x) \rangle_{\mathbb{C}^d}, \quad \forall n \in \mathbb{N}.$$

Since the right-hand function is integrable, the Dominated Convergence Theorem implies that

$$\lim_{n \rightarrow \infty} \|f\|_{L^2(W_n)}^2 = \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \langle W_n(x)f(x), f(x) \rangle_{\mathbb{C}^d} dx = \|f\|_{L^2(W)}^2.$$

To obtain (17), first observe that

$$\begin{aligned} \sum_{I \in \mathcal{D}} \left\langle \langle W_n \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} &= \frac{1}{n} \sum_{I \in \mathcal{D}} \left\langle \langle P_{E_1^n(x)} \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &\quad + \sum_{I \in \mathcal{D}} \left\langle \langle P_{E_2^n(x)} W(x) P_{E_2^n(x)} + nP_{E_3^n(x)} \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d}. \end{aligned}$$

The first term clearly goes to zero as $n \rightarrow \infty$. Meanwhile, the terms in the second sum are increasing in n . So, we can apply the Monotone Convergence Theorem twice to conclude

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{I \in \mathcal{D}} \left\langle \langle W_n \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} &= \sum_{I \in \mathcal{D}} \lim_{n \rightarrow \infty} \left\langle \langle P_{E_2^n(x)} W(x) P_{E_2^n(x)} + n P_{E_3^n(x)} \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \left\langle \left\langle \lim_{n \rightarrow \infty} P_{E_2^n(x)} W(x) P_{E_2^n(x)} + n P_{E_3^n(x)} \right\rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \hat{f}(I), \hat{f}(I) \right\rangle_{\mathbb{C}^d}. \end{aligned}$$

Now, letting $n \rightarrow \infty$ in (15) and using (14), (16), and (17) gives Theorem 3.2 for general W , since $L^2(W) \cap L^2$ is dense in $L^2(W)$. Theorem 3.1 follows similarly.

4. THE HILBERT TRANSFORM

The bounds given in Theorems 3.1 and 3.2 imply similar bounds for the Hilbert transform on $L^2(W)$. To see this, fix $\alpha \in \mathbb{R}$ and $r > 0$. The densely-defined shift operator $\mathbb{H}^{\alpha,r}$ on $L^2(\mathbb{R}, \mathbb{C})$ is given by

$$\mathbb{H}^{\alpha,r} f \equiv \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{D}^{\alpha,r}} \hat{f}(I) (h_{I_-} - h_{I_+}).$$

In [27], Petermichl showed that the Hilbert transform H on $L^2(\mathbb{R}, \mathbb{C})$ is basically an average of these dyadic shifts. Specifically, there is a constant c and $L^\infty(\mathbb{R}, \mathbb{C})$ function b such that $H = cT + M_b$, where T is in the weak operator closure of the convex hull of the set $\{\mathbb{H}^{\alpha,r}\}_{\alpha,r}$ in $\mathcal{L}(L^2(\mathbb{R}, \mathbb{C}))$ and M_b is multiplication by b . The Hilbert transform on $L^2(\mathbb{R}, \mathbb{C}^d)$, also denoted H , is the scalar Hilbert transform applied component-wise. The dyadic shift operators $\mathbb{H}^{\alpha,r}$ on $L^2(\mathbb{R}, \mathbb{C}^d)$ are similarly defined by

$$\mathbb{H}^{\alpha,r} f \equiv \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{D}^{\alpha,r}} \hat{f}(I) (h_{I_-} - h_{I_+}),$$

which is the same as applying the scalar $\mathbb{H}^{\alpha,r}$ shifts component-wise. Using the scalar-result, the Hilbert transform H on $L^2(\mathbb{R}, \mathbb{C}^d)$ satisfies $H = c\tilde{T} + M_b$ where \tilde{T} is T applied component-wise and so, is in the weak operator closure of the convex hull of the set $\{\mathbb{H}^{\alpha,r}\}_{\alpha,r}$ in $\mathcal{L}(L^2(\mathbb{R}, \mathbb{C}^d))$.

In [41], Treil and Volberg showed that for A_2 weights W , the Hilbert transform is bounded on $L^2(W)$, but they did not track the dependence on the A_2 characteristic $[W]_{A_2}$. In contrast, using our square function estimates, we are able to establish the following:

Theorem 4.1. *Let W be a $d \times d$ matrix weight in A_2 . Then*

$$\|Hf\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2} \|f\|_{L^2(W)} \quad \forall f \in L^2(W).$$

Proof. As before, we omit the α, r notation. Observe that the square function norm in Theorems 3.1 and 3.2 is not affected by dyadic shifts. Specifically, let \tilde{I} denote the parent

of I in the dyadic grid. Then

$$\begin{aligned}
\|S_W \mathbb{I} f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{\mathbb{I} f}(I), \widehat{\mathbb{I} f}(I) \right\rangle_{\mathbb{C}^d} \\
&= \frac{1}{2} \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(\tilde{I}), \widehat{f}(\tilde{I}) \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I \in \mathcal{D}} \left\langle \frac{1}{2} (\langle W \rangle_{I_-} + \langle W \rangle_{I_+}) \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\
&= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\
&= \|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2.
\end{aligned}$$

Now, using Theorems 3.1 and 3.2, we have

$$\begin{aligned}
\|\mathbb{I} f\|_{L^2(W)}^2 &\lesssim [W]_{A_2} \log [W]_{A_2} \|S_W \mathbb{I} f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \\
&= [W]_{A_2} \log [W]_{A_2} \|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \\
&\lesssim [W]_{A_2}^3 (\log [W]_{A_2})^2 \|f\|_{L^2(W)}^2.
\end{aligned}$$

The formula for H in terms of dyadic shifts implies that

$$\|Hf\|_{L^2(W)}^2 \lesssim \sup_{\alpha, r} \|\mathbb{I}^{\alpha, r} f\|_{L^2(W)}^2 + \|b\|_{\infty}^2 \|f\|_{L^2(W)}^2 \lesssim [W]_{A_2}^3 (\log [W]_{A_2})^2 \|f\|_{L^2(W)}^2,$$

as desired. \square

5. HAAR MULTIPLIERS

The arguments above extend easily to Haar multipliers. To begin, let $\sigma = \{\sigma_I\}_{I \in \mathcal{D}}$ be a sequence of $d \times d$ matrices and recall the Haar multiplier T_{σ} defined by

$$T_{\sigma} f \equiv \sum_{I \in \mathcal{D}} \sigma_I \widehat{f}(I) h_I.$$

To obtain boundedness on $L^2(W)$, it is crucial that the matrices σ_I interact well with W . To be precise, fix a weight $W \in A_2$ and define

$$\|\sigma\|_{\infty} \equiv \inf \left\{ C : \langle W \rangle_I^{-\frac{1}{2}} \sigma_I^* \langle W \rangle_I \sigma_I \langle W \rangle_I^{-\frac{1}{2}} \leq C^2 I_{d \times d} \quad \forall I \in \mathcal{D} \right\}.$$

Equivalently, we could define $\|\sigma\|_{\infty} = \sup_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{\frac{1}{2}} \sigma_I \langle W \rangle_I^{-\frac{1}{2}} \right\|$. Then, a variant of the following result is established by Isralowitz-Kwon-Pott in [18]:

Theorem 5.1. *Let $W \in A_2$ and $\sigma = \{\sigma_I\}_{I \in \mathcal{D}}$ a sequence of matrices. Then the Haar multiplier T_{σ} is bounded on $L^2(W)$ if and only if $\|\sigma\|_{\infty} < \infty$.*

Here, we have translated their result to the notation of this paper. Now, we provide a new and simpler proof of this boundedness result for $p = 2$. Using our previous arguments, we are also able to track the dependence on $[W]_{A_2}$.

Theorem 5.2. *Let W be a $d \times d$ matrix weight in A_2 and let $\sigma = \{\sigma_I\}_{I \in \mathcal{D}}$ be a sequence of $d \times d$ matrices. Then T_{σ} is bounded on $L^2(W)$ if and only if $\|\sigma\|_{\infty} < \infty$. Moreover,*

$$\|T_{\sigma} f\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log [W]_{A_2} \|\sigma\|_{\infty} \|f\|_{L^2(W)}.$$

Proof. Necessity is almost immediate. Fix $I \in \mathcal{D}$ and $e \in \mathbb{C}^d$ and simply set $f \equiv \langle W \rangle_I^{-\frac{1}{2}} h_I e$. Then simple computations prove that $T_\sigma f = \sigma_I \langle W \rangle_I^{-\frac{1}{2}} h_I e$ and the following norm equalities:

$$\begin{aligned} \|f\|_{L^2(W)}^2 &= \|\langle W \rangle_I^{-\frac{1}{2}} h_I e\|_{L^2(W)}^2 = \|e\|_{\mathbb{C}^d}^2; \\ \|T_\sigma f\|_{L^2(W)}^2 &= \left\| \sigma_I \langle W \rangle_I^{-\frac{1}{2}} h_I e \right\|_{L^2(W)}^2 = \left\langle \langle W \rangle_I^{-\frac{1}{2}} \sigma_I^* \langle W \rangle_I \sigma_I \langle W \rangle_I^{-\frac{1}{2}} e, e \right\rangle_{\mathbb{C}^d}. \end{aligned}$$

Assuming T_σ is bounded on $L^2(W)$, we can then conclude:

$$\left\langle \langle W \rangle_I^{-\frac{1}{2}} \sigma_I^* \langle W \rangle_I \sigma_I \langle W \rangle_I^{-\frac{1}{2}} e, e \right\rangle_{\mathbb{C}^d} = \|T_\sigma f\|_{L^2(W)}^2 \leq \|T_\sigma\|_{L^2(W) \rightarrow L^2(W)}^2 \|e\|_{\mathbb{C}^d}^2.$$

Since $e \in \mathbb{C}^d$ and $I \in \mathcal{D}$ was arbitrary we have that $\|\sigma\|_\infty < \infty$.

The proof of sufficiency, with the desired constant, is largely a repetition of computations from earlier in the paper. As before, observe that the square function in Theorems 3.1 and 3.2 interacts well with Haar multipliers. Specifically,

$$\begin{aligned} \|S_W T_\sigma f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{T_\sigma f}(I), \widehat{T_\sigma f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \sigma_I \widehat{f}(I), \sigma_I \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \sigma_I \langle W \rangle_I^{-\frac{1}{2}} \langle W \rangle_I^{\frac{1}{2}} \widehat{f}(I), \sigma_I \langle W \rangle_I^{-\frac{1}{2}} \langle W \rangle_I^{\frac{1}{2}} \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &\leq \|\sigma\|_\infty^2 \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I \widehat{f}(I), \widehat{f}(I) \right\rangle_{\mathbb{C}^d} \\ &= \|\sigma\|_\infty^2 \|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2. \end{aligned}$$

Simple applications of Theorems 3.1 and 3.2 then yield

$$\begin{aligned} \|T_\sigma f\|_{L^2(W)}^2 &\lesssim [W]_{A_2} \log [W]_{A_2} \|S_W T_\sigma f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \\ &\leq [W]_{A_2} \log [W]_{A_2} \|\sigma\|_\infty^2 \|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \\ &\lesssim [W]_{A_2}^3 (\log [W]_{A_2})^2 \|\sigma\|_\infty^2 \|f\|_{L^2(W)}^2, \end{aligned}$$

which gives the desired bound. \square

Remark 5.3. One should observe that the arguments in Theorems 4.1 and 5.2 rest on a good relationship between the operator in question and the square function S_W . Specifically, in Theorem 4.1, the family of dyadic shifts $\text{III}^{\alpha, r}$ interacts well with S_W and by taking averages of them, one can recover the Hilbert transform. It is not hard to show that S_W also interacts well with other similarly-nice dyadic shifts, which one could use to build other Calderón-Zygmund operators and obtain similar estimates.

6. OPEN QUESTIONS

6.1. Square Function Estimates. If w is a scalar-valued A_2 weight, then Theorem 3.1 is true with $[w]_{A_2}^2$ replacing $[w]_{A_2}^2 \log [w]_{A_2}$. This motivates the following conjecture:

Conjecture 6.1. *Let W be a $d \times d$ matrix weight in A_2 . Then*

$$\|S_W f\|_{L^2(\mathbb{R}, \mathbb{R})}^2 \lesssim [W]_{A_2}^2 \|f\|_{L^2(W)}^2 \quad \forall f \in L^2(W).$$

To prove Conjecture 6.1, we would need to control the term S_3 from (12) in a more optimal way. Our current method of using Theorem 2.1 introduces the troublesome $\log[W]_{A_2}$ term. An alternate method of controlling S_3 would use a matrix version of the weighted Carleson Embedding Theorem. One would first control S_3 by

$$\begin{aligned} S_3 &= \sum_{I \in \mathcal{D}} \sum_{k=1}^d \left| \left\langle \langle W \rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W \rangle_I^{-1} \langle f \rangle_I, e_I^k \right\rangle_{\mathbb{C}^d} \right|^2 \\ &\lesssim \sum_{I \in \mathcal{D}} \left\| \langle W \rangle_I^{-\frac{1}{2}} \widehat{W}(I) \langle W \rangle_I^{-1} \langle f \rangle_I \right\|_{\mathbb{C}^d}^2 \\ &= \sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1} \langle f \rangle_I, \langle f \rangle_I \right\rangle_{\mathbb{C}^d}. \end{aligned}$$

Conjecture 6.1 would follow if we could show

$$\sum_{I \in \mathcal{D}} \left\langle \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1} \langle f \rangle_I, \langle f \rangle_I \right\rangle_{\mathbb{C}^d} \lesssim [W]_{A_2} \|f\|_{L^2(W^{-1})}^2.$$

To obtain this, we need two things. First, we need a matrix version of the weighted Carleson Embedding Theorem. Very recently, the needed result has actually been proven by Culiuc-Treil in [6], who show the following:

Theorem 6.2. *Let W be a $d \times d$ matrix weight and let $\{A_I\}_{I \in \mathcal{D}}$ be a sequence of positive semidefinite $d \times d$ matrices indexed by the dyadic intervals. Then*

$$\sum_{I \in \mathcal{D}} \langle A_I \langle f \rangle_I, \langle f \rangle_I \rangle_{\mathbb{C}^d} \lesssim C \|f\|_{L^2(W^{-1})}^2 \quad \forall f \in L^2(W^{-1})$$

if and only if

$$\frac{1}{|J|} \sum_{I \subset J} \langle W \rangle_I A_I \langle W \rangle_I \leq C \langle W \rangle_J \quad \forall J \in \mathcal{D}.$$

Second, we need the appropriate testing conditions to apply Theorem 6.2. Specifically The A_I that appear in our bound for S_3 are the matrices

$$\langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1} \widehat{W}(I) \langle W \rangle_I^{-1}.$$

Given Theorem 6.2, we need the appropriate testing condition to apply it to S_3 . Indeed, we require

$$(18) \quad \frac{1}{|J|} \sum_{I \subset J} \widehat{W}(I) \langle W \rangle_I^{-1} \widehat{W}(I) \lesssim [W]_{A_2}^2 \langle W \rangle_J, \quad \forall J \in \mathcal{D}.$$

In the scalar case, this is proved by Wittwer in [43] using estimates from Buckley [3]. Hukovic-Treil-Volberg give a Bellman function proof in [13]. Neither of these arguments adapt well to the matrix setting and currently, it is not clear whether (18) is true for matrices.

6.2. The Hilbert Transform and Haar Multipliers. If w is a scalar A_2 weight, then Theorems 4.1 and 5.2 are true with $[w]_{A_2}$ replacing $[w]_{A_2}^{\frac{3}{2}} \log[w]_{A_2}$. This motivates the following conjecture:

Conjecture 6.3. *Let W be a $d \times d$ matrix weight in A_2 and let $\{\sigma_I\}_{I \in \mathcal{D}}$ be a sequence of matrices satisfying $\|\sigma\|_\infty < \infty$. Then*

$$\begin{aligned} \|H\|_{L^2(W) \rightarrow L^2(W)} &\lesssim [W]_{A_2}; \\ \|T_\sigma\|_{L^2(W) \rightarrow L^2(W)} &\lesssim [W]_{A_2} \|\sigma\|_\infty. \end{aligned}$$

Given our current tools, those estimates seem out of reach. However, if we could prove Conjecture 6.1 by establishing a bound of $[W]_{A_2}$ in Theorem 3.2, then the arguments from the proofs of Theorems 4.1 and 5.2 would immediately imply that

$$\begin{aligned} \|H\|_{L^2(W) \rightarrow L^2(W)} &\lesssim [W]_{A_2}^{\frac{3}{2}}; \\ \|T_\sigma\|_{L^2(W) \rightarrow L^2(W)} &\lesssim [W]_{A_2}^{\frac{3}{2}} \|\sigma\|_\infty. \end{aligned}$$

6.3. Extrapolation. In the scalar case, after Hytönen established

$$\|T\|_{L^2(w) \rightarrow L^2(w)} \lesssim [w]_{A_2},$$

for all Calderón-Zygmund operators T , he used the sharp form of Rubio de Francia's extrapolation theorem due to Dragičević-Grafakos-Pereyra-Petermichl [7] to obtain the following

$$\|T\|_{L^p(w) \rightarrow L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \quad 1 < p < \infty,$$

which is sharp for all exponents. For the dyadic square function, one can also extrapolate weighted L^p bounds from the weighted L^2 bounds, but the estimates are only sharp for $1 < p \leq 2$. These extrapolation results rely heavily on maximal functions; Rubio de Francia's original extrapolation theorem [38] used the connections between the maximal function and A_p weights. Similarly, the sharp theorem in [7] used the sharp dependence of the maximal function's norm on $[w]_{A_p}$:

$$\|M\|_{L^p(w) \rightarrow L^p(w)} \lesssim [w]_{A_p}^{\frac{p'}{p}} \quad 1 < p < \infty,$$

first proved by Buckley in [4]. Then, a natural question is:

Can one use extrapolation and the operator bounds from Theorems 3.1, 4.1, and 5.2 to deduce similar bounds for operators related to A_p weights, $1 < p < \infty$?

This is an interesting question certainly worth exploring. However, there are several complications stemming from maximal functions in the vector-valued setting, which make the question difficult. First, a problem arises when defining maximal operators in the vector case. For example, one could define $Mf(x)$ to be the average of f over an interval containing x with largest vector magnitude. However, this ignores the fact that the effect of a matrix weight W will depend both on direction and magnitude. Instead, in [5, 10], Christ-Goldberg and Goldberg studied weighted, vector analogues of the maximal function and defined a different operator M_W^p for each weight W and $1 < p < \infty$. For the exact formulas, see (9) in Section 3. The boundedness of these maximal operators is closely related to the weight W belonging to a specific A_p class. However, proving related extrapolation results seems difficult because we now have a family of maximal

operators that rely on both the weight W and the exponent p . Furthermore, although Isralowitz-Kwon-Pott in [18] established the nice sharp bound

$$\|M_W^2\|_{L^2(\mathbb{R}, \mathbb{C}^d) \rightarrow L^2(\mathbb{R}, \mathbb{R})} \lesssim [W]_{A_2},$$

the sharp bounds for $p \neq 2$ are currently unknown. It is worth noting that Isralowitz-Kwon-Pott indicate in [18] that similar bounds for $p \neq 2$ will be established in [20].

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